Equivalent Statements

Two statements are logically equivalent if both statements have exactly the same truth values in the answer column of their truth tables. We use the symbols $\Leftrightarrow$ or $\equiv$ for logical equivalence.

Example:
Determine whether the following two statements are logically equivalent.
$\sim p \vee \sim q$ and $\sim(p \vee q)$

p	q	$\sim p$	$\vee$	$\sim q$	$\sim$	$(p$	$\wedge$	$q)$
T	T	F	F	F	F	T	T	T
T	F	F	T	T	T	T	F	F
F	T	T	T	F	T	F	F	T
F	F	T	T	T	T	F	F	F

Steps:
1. Find the truth value of the first statement
    a. Step 1: red
    b. Step 2: compare two columns of red truth values to arrive at the truth value for the whole statement (shown in purple).
2. Find the truth value of the second statement
    a. Step 1: green
    b. Step 2: compare the two columns of green truth values to arrive at the truth value for the statement in parentheses (shown in blue).
    c. Step 3: Find the negative (opposite) of the truth values from the previous step (shown in purple).
3. The answer columns for the two statements are shown in purple. Compare these columns. If they look exactly alike, then the statements are logically equivalent. Note: It only takes on truth value being different to keep the statements from being logically equivalent.

$\therefore$ The statements are logically equivalent, because their truth tables are exactly the same.
Thus, $\sim p \vee \sim q \Leftrightarrow \sim(p \vee q)$

This particular example covers one of a special set of laws that will help us determine the logical equivalence of statements. These laws are called the DeMorgan's Laws, named after Augustus DeMorgan and English mathematician.

DeMorgan's Laws

1. $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$
2. $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

You can use these laws to recognize whether two statements are equivalent without having to build their truth tables. However, for this class I will require you to build the truth tables most of the time.

Example:

Show that $p \rightarrow q \Leftrightarrow \sim p \vee q$

Recall that $p \rightarrow q$ is true in every case except when p = True and q = False (in otherwords, the broken promise).

p	q	p	$\rightarrow$	q	$\sim p$	$\vee$	q
T	T	T	T	T	F	T	T
T	F	T	F	F	F	F	F
F	T	F	T	T	T	T	T
F	F	F	T	F	T	T	F

Step 1 in red.
Step 2 in purple.
Step 3 in green.
Step 4 in blue.

Now compare the answer columns (one is in purple and the other is in blue). The statements have exactly the same truth values. Therefore, the statements are logically equivalent.

$\therefore p \rightarrow q \Leftrightarrow \sim p \vee q$

Negation of the Conditional Statement

In the example above, we showed that $p \rightarrow q \Leftrightarrow \sim p \vee q$. Now let's use that knowledge to determine what $\sim(p \rightarrow q)$ is equivalent to.

So if

$p \rightarrow q \Leftrightarrow \sim p \vee q$

Then

$\sim(p \to q) \Leftrightarrow \sim(\sim p \vee q)$

(All we did here was replace the statement in the parentheses $p \to q$ with its equivalent form $\sim p \vee q$).

Now we will destribute the negation that is on the outside of the parentheses (on the right side of the $\Leftrightarrow$). So we have the following:

$$\begin{aligned} \sim(p \to q) &\Leftrightarrow \sim(\sim p \vee q) \\ &\Leftrightarrow \sim(\sim p) \wedge \sim q \end{aligned}$$

Next, we simplify the statement on the right of the $\Leftrightarrow$ as much as possible.

$$\begin{aligned} \sim(p \to q) &\Leftrightarrow \sim(\sim p \vee q) \\ &\Leftrightarrow \sim(\sim p) \wedge \sim q \\ &\Leftrightarrow p \wedge \sim q \end{aligned}$$

Thus, $\sim(p \to q) \Leftrightarrow p \wedge \sim q$

This is just one of the several variations of the conditional statement. Each has a special name.

Name	Symbolic Form	Read as:
Conditional	$p \to q$	"If p, then q"
Converse	$q \to p$	"If q, then p"
Inverse	$\sim p \to \sim q$	"If not p, then not q"
Contrapositive	$\sim q \to \sim p$	"If not q, then not p"

In other words:

Converse: switch the p and q

Inverse: negate both p and q

Contrapositive: negate and switch bot p and q

<u>Example</u>: Determine whether any of the forms of the conditional statement are logically equivalent.

(i.e. Construct the truth tables)

p	q	$p \to q$	q	$\to$	p	$\sim p$	$\to$	$\sim q$	$\sim q$	$\to$	$\sim p$
T	T	T	T	T	T	F	T	F	F	T	F
T	F	F	F	T	T	F	T	T	T	F	F
F	T	T	T	F	F	T	F	F	F	T	T
F	F	T	F	T	F	T	T	T	T	T	T

Therefore, $p \to q \Leftrightarrow (\sim q \to \sim p)$ and $q \to p \Leftrightarrow (\sim p \to \sim q)$.

Examples: Use DeMorgan's Laws to determine whether the statements are equivalent.

#12 $\sim (p \wedge q)$, $(p \vee \sim q)$

Start with one of the two statements and try to work towards the other through manipulations and simplifications.

$$\begin{aligned} \sim (p \wedge q) &\Leftrightarrow \sim p \vee \sim q \\ &\not\Leftrightarrow p \vee \sim q \end{aligned}$$

Thus, these two statements are not equivalent.

Note: An alternate method would be to show the truth tables for both statements to determine equivalence.

#18 $\sim (\sim p \to q)$, $\sim p \wedge \sim q$

Recall that $g \to h \Leftrightarrow \sim g \vee h$.

Start with the first statement and try to manipulate it to look like the second.

$$\begin{aligned} \sim (\sim p \to q) &\Leftrightarrow \sim (\sim (\sim p) \vee q) \\ &\Leftrightarrow \sim (p \vee q) \\ &\Leftrightarrow \sim p \wedge \sim q \end{aligned}$$

Therefore, the statements are logically equivalent.